BADMC Solutions

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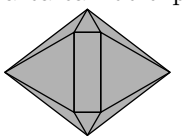
1 Problems

1. Find $\frac{1}{0.1} + \frac{2}{0.2} + ... + \frac{9}{0.9}$
   Each of these terms can be expressed as $\frac{n}{\frac{n}{10}} = \frac{1}{\frac{1}{10}} = 10$, so because there are 9 total terms, the total sum is $\boxed{90}$.

2. If all triangles are equilateral and the rectangle is 10 by 30, the shaded area can be expressed as $a + b\sqrt{c}$. Find $a + b + c$.

   Note that the shaded area is a rhombus. The horizontal side length is $10 + 15\sqrt{3} + 15\sqrt{3} = 10 + 30\sqrt{3}$, and the vertical side length is $30 + 5\sqrt{3} + 5\sqrt{3} = 30 + 10\sqrt{3}$. Multiplying these and dividing by two gives $600 + 500\sqrt{3}$, so the answer is $600 + 500 + 3 = \boxed{1103}$

3. Find $\sum_{x=-1000}^{1000} \lfloor \sqrt[3]{x} \rfloor$
   If x is positive and not a cube, then $\sqrt[3]{-x} = -\sqrt[3]{x}$, so $\lfloor \sqrt[3]{-x} \rfloor = -\lfloor \sqrt[3]{x} \rfloor - 1$. $\lfloor \sqrt[3]{-x} + \lfloor \sqrt[3]{x} \rfloor = -1$. However, when x is a perfect cube, $\lfloor \sqrt[3]{-x} \rfloor = -\lfloor \sqrt[3]{x} \rfloor$, so $\lfloor \sqrt[3]{-x} \rfloor + \lfloor \sqrt[3]{x} \rfloor = 0$. From 1 to 1000, there are 10 perfect cubes $(1^3, 2^3, ..., 10^3)$, so there are 990 non-perfect cubes. So the answer is $\boxed{-990}$.

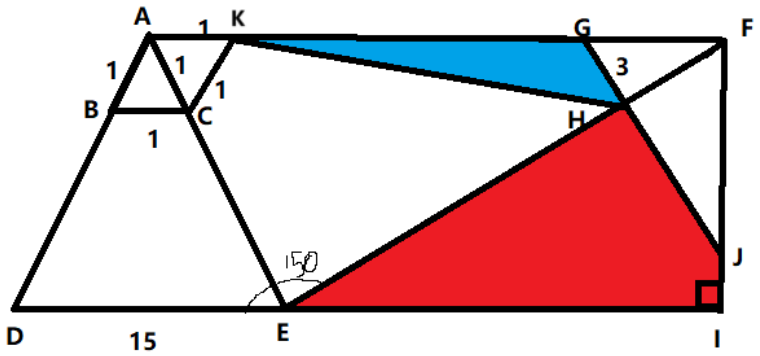
4. Find the sum of all values of n such that $\sqrt{n^{n^n}}$ is an integer and $1 \leq n \leq 100$
   Any even n works because the exponent (n^{n^n}) is even. Any odd perfect squares n also work since the exponent (n^{n^n}) is odd, so when divided by two, it will end with 0.5. A perfect square to the power of 0.5 results in an integer, so odd perfect squares work. The sum of all even values between 1 and 100 is $2+4+6+...+100 = 2(1+2+3+...+50) = 2(\frac{50*51}{2}) = 2550$. The sum of all the odd squares is $1^2 + 3^2 + 5^2 + 7^2 + 9^2 = 165$. So the total sum is $2550 + 165 = \boxed{2715}$.

5. Find $\gcd(95040, 9995000400, 999995000004000)$.
Let $x = 10$. $95040 = x^5 - x^4 + 5x^3 + 4x = x(x^4 - 5x^2 + 4) = x(x^2 - 4)(x^2 - 1) = x(x+2)(x-2)(x+1)(x-1) = (x-2)(x-1)x(x+1)(x+2)$, so this can be factored as $2^6 * 3^3 * 5 * 11$. For the second term, it also equals $(x-2)(x-1)x(x+1)(x+2)$, but $x = 100$. This can be factored as $2^4 * 3^3 * 5^2 * 7^2 * 11 * 17 * 101$. Similarly, the last term equals $998 * 999 * 1000 * 1001 * 1002 = 2^5 * 3^4 * 5^3 * 7 * 11 * ...$, so the GCD becomes $2^4 * 3^3 * 5 * 11 = \frac{95040}{4} = \boxed{23760}$

6. Order the numbers below in increasing order. Write your answer as a permutation of 1234.
1) 0.5^{100}
2) 0.12^{30}
3) 0.9^{314}
4) 0.2^{51}

$0.5^{100} = (0.5^{10})^{10} = (\frac{1}{1024})^{10}$
$0.12^{30} = 0.001728^{10} > 0.001^{10} > (\frac{1}{1024})^{10} = 0.5^{100}$
$0.9^{314} = 0.9^{14} * 0.9^{300}$. The 0.9^{14} is very close to 1 and can be ignored.
$0.9^{300} = (0.9^{10})^{30} > (0.8^5)^{30} = (0.4096 * 0.8)^{30} > (0.32)^{30} > 0.12^{30}$
$0.2^{51} < 0.2^{50} = \frac{1}{\sqrt{5}}^{100} < \frac{1}{2}^{100} = 0.5^{100}$
Using these get that the correct order is $\boxed{4123}$

7. If $BC \parallel DE$ and $GJ \parallel AE$, the sum of the areas of the blue and red regions in the diagram below can be expressed as $\frac{a\sqrt{b}}{c}$ where b is square-free and $\gcd(a, c) = 1$. Find $a + b + c$.



Because $\triangle ABC$ is equilateral, $\triangle ADE$ is also equilateral and $AE = 15$. Also, $\angle AED = 60$, so $\angle AEF = 90$. Because $\triangle AKC$ is equilateral, $\angle CAK = 60$ so $\triangle AEF$ is a $30 - 60 - 90$ triangle and $AF = 30$. Because $\angle GFH = 30$, $\triangle GHF$ is also $30-60-90$, so $GF = 6$ and $KG = 30-1-6 =$

23. By the sine area formula, the blue area is $\frac{23*3*\sin 120}{2} = \frac{69\sqrt{3}}{4}$. Also, HFJ is $30-60-90$, so because $HF = 3\sqrt{3}$, $HJ = 9$, and the area of HFJ is $\frac{27\sqrt{3}}{2}$. $EI = AF - \frac{15}{2} = \frac{45}{2}$ and $FI = \frac{15\sqrt{3}}{2}$ (the altitude of ADE). So the area of $\triangle EIF$ is $\frac{675\sqrt{3}}{8}$ and the area of $EHJI$ is $\frac{675\sqrt{3}}{8} - \frac{27\sqrt{3}}{2} = \frac{567\sqrt{3}}{8}$. Adding this to the blue area gives $\frac{705\sqrt{3}}{8}$ and the answer is $705+3+8=\boxed{716}$

8. How many factors of 10! have a units digit of 1?
$10! = 2^8 * 3^4 * 5^2 * 7$. The units digit is 1 if there are no 2s and 5s in a factor. So it can only have 3s and 7s. The unit digits of the five possible powers of threes are $1, 3, 9, 7, 1$. When each of them are multiplied by 7, the units digits become $7, 1, 3, 9, 7$. There are 3 ones, so the answer is $\boxed{3}$.

9. Find the area of a triangle with side lengths $\sqrt{53}, \sqrt{13}, 5\sqrt{2}$ rounded to the nearest tenth.
The triangle with coordinates $(3,0), (5,7), (0,2)$ satisfies this. Use the shoelace formula to get that the area is $\boxed{12.5}$.

10. There are 45 weights with $6, 10, 15$, and 30 pounds (180 weights in total). How many ways are there to place these weights on a scale so the total weight is 300 pounds?
Let a, b, c, d be the number of weights with masses of $6, 10, 15, 30$, respectively. $6a+10b+15c+30d = 300$. Take mod 3 to get that $b \equiv 0 \pmod 3$. Let $b = 3e$. $6a + 30e + 15c + 30d = 300$. $2a + 10e + 5c + 10d = 100$. Take mod 5 to get that $2a \equiv 0 \pmod 5$. Let $a = 5f$. $10f+10e+5c+10d = 100$. $2f + 2e + c + 2d = 20$.Take mod 2 to get that $c \equiv 0 \pmod 2$. Let $c = 2g$. $2f + 2e + 2g + 2d = 20$. $f + e + g + d = 10$. If there are infinite amount of weights, this has $\binom{10+4-1}{4-1} = 286$ ways. But it is possible for a to be more than 45. Because a has to be divisible by 5, if it is equal to 50, then $b = c = d = 0$ works, so minus one to get $\boxed{285}$.

11. The numbers $187, 665, 547, 069$, and 420 are concatenated in a specific order to form a number n divisible by 13. If n can start with a 0, find the smallest possible value of n.
Note that because $1000 \equiv -1 \pmod 1 3$, and $n = 1000^4 * a + 1000^3 * b + ... + e$ where a, b, c, d, e are arrangements of $187, 665, 547, 069, 420$, $n \pmod 1 3 \equiv a - b + c - d + e = (a+b+c+d+e) - 2b - 2d \equiv 3 - 2b - 2d \equiv 0$. Taking each of the numbers mod 13 results in $5, 2, 1, 4, 4$, and $3 - 2b - 2d \equiv 0 \pmod 1 3$ means that $2b + 2d \equiv 3 \equiv 16 \pmod 1 3$. So $b = d = 4$ works, and $\boxed{187069547420665}$ is the smallest possible value of n.

12. Find the smallest integer n such that $n! > 5^n$ For $n = 4$, get that $\frac{5^n}{n!} \approx 26$ For $n = 5$, this does not change because the numerator and denominator are both multiplied by five. Later for $n = m$, the fraction is multiplied by $\frac{5}{m}$. So $26 * \frac{5}{5} * \frac{5}{6} * ... * \frac{5}{m} < 1$
$\frac{5^{m-4}}{5(6)(7)...(m)} < \frac{1}{26}$

$5(6)(7)...(m) > 26 * 5^{m-4}$
The average term of the left side product is $\frac{5+m}{2}$. So it can be estimated as $(\frac{5+m}{2})^{m-4} > 26 * 5^{m-4}$
Testing $m = 11$ results in the inequality becoming $2^{21} > 26 * 5^7 = 26 * 5 * (5^3)^2 > 130 * 120^2 = 1872000$, but $2^{21} = 2 * (2^{10})^2 \approx 2 * 1000^2 = 2000000$, which is satisfied. But plugging $m = 11$ back into the original equation results in $26 * 5^{m-4}$ being slightly larger (less than 12/5), so multiplying by 5/12 makes the $5 * 6 * 7 * ... * m$ larger, and the answer is $n = \boxed{12}$